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Three kinds of convergence and the associated $\mathcal{I}$-Baire classes ☆

Rafał Filipów, Piotr Szuca *

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

We consider ideal versions of pointwise, discrete and equal convergence of sequences of functions. Defining, in a natural way, ideal pointwise (discrete, equal) Baire classes of functions, we show that these classes are equal to their classical counterparts for ideals for which there is a winning strategy in a game introduced by Laflamme (1996) [10]. In the proofs we make extensive use of a characterization (in terms of filters $\mathcal{F}$ which are ω-diagonalizable by $\mathcal{F}$-universal sets) of a winning strategy. This article extends results of Laczkovich and Recław (2009) [9], and Debs and Saint Raymond (2009) [5].

1. Introduction

The set of all natural numbers is denoted by ω. A family of sets $\mathcal{I} \subset \mathcal{P}(\omega)$ is an *ideal* if it is closed under taking finite unions and subsets. Throughout this paper we assume that $\mathcal{I}$ contains all finite sets and $\omega \notin \mathcal{I}$. We can talk about ideals on any countable set by identifying this set with ω via a fixed bijection. The family $\mathcal{I}^\star = \{\omega \setminus A\colon\ A \in \mathcal{I}\}$ is a *filter*, i.e., a family of sets closed under taking finite intersections and supersets. If $\mathcal{F}$ is a filter then $\mathcal{F}^\star = \{\omega \setminus A\colon\ A \in \mathcal{F}\}$ is an ideal. For a filter $\mathcal{F} = \mathcal{I}^\star$, $\mathcal{F}^+ = \mathcal{I}^+ = \{A \subset \omega\colon\ A \notin \mathcal{I}\}$.

A sequence of reals $(x_n)_{n\in\omega}$ is *$\mathcal{I}$-convergent* to $x \in \mathbb{R}$ if for every $\varepsilon > 0$ $\{n \in \omega\colon\ |x - x_n| \geqslant \varepsilon\} \in \mathcal{I}$. We write $\mathcal{I} - \lim x_n = x$ if $(x_n)_{n\in\omega}$ is $\mathcal{I}$-convergent to x. We say that a sequence of functions $(f_n)_{n\in\omega}$ ($f_n : X \to \mathbb{R}$) is *$\mathcal{I}$-pointwise convergent* to $f : X \to \mathbb{R}$ if $\mathcal{I} - \lim f_n(x) = f(x)$ for every $x \in X$.

The game $G(\mathcal{I})$ is defined as follows: player I in the nth move plays an element $C_n \in \mathcal{I}$, and then player II plays a finite set $F_n \subset \omega$ with $F_n \cap C_n = \emptyset$. Player I wins when $\bigcup_n F_n \in \mathcal{I}$, otherwise player II wins. This game was investigated by Laflamme [10], who denoted it by $\mathfrak{G}(\mathcal{I}^\star, [\omega]^{<\omega}, \mathcal{I}^+)$.

Laczkovich and Recław proved [9, Prop. 8] that if $\mathcal{I}$ is an ideal such that player II has a winning strategy in $G(\mathcal{I})$ and X is a complete metric space, then every $\mathcal{I}$-pointwise limit of a sequence of continuous functions is of the first Baire class.

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* Corresponding author.
E-mail addresses: rfilipow@mat.ug.edu.pl (R. Filipów), pszuca@radix.com.pl (P. Szuca).
URL: http://rfilipow.mat.ug.edu.pl/ (R. Filipów).

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A set $\mathcal{Z} = \{A_m\colon m \in \omega\} \subset [\omega]^{<\omega} \setminus \{\emptyset\}$ is *$\mathcal{I}^\star$-universal* if for each $F \in \mathcal{I}^\star$ there is an $m \in \omega$ such that $A_m \subset F$. We say that $\mathcal{I}^\star$ is *ω-diagonalizable by $\mathcal{I}^\star$-universal sets* if there exists a sequence $(\mathcal{Z}_N)_{N\in\omega}$ of $\mathcal{I}^\star$-universal sets such that for each $F \in \mathcal{I}^\star$ there is $\mathcal{Z}_N = \{A_{N,m}\colon m \in \omega\}$ such that $(\exists M \in \omega)\ (\forall m > M)\ (A_{N,m} \cap F \neq \emptyset)$.

Laflamme proved [10, Thm. 2.16] that player II has a winning strategy in $G(\mathcal{I})$ if and only if $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Thus, the result of Laczkovich and Recław can be reformulated as follows: if X is a complete metric space and $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets, then every $\mathcal{I}$-pointwise limit of a sequence of continuous functions is of the first Baire class. Using this characterization of a winning strategy for player II, we extend the result of Laczkovich and Recław to all perfectly normal topological spaces (Theorem 3.2).

In [9, Prop. 10 and Prop. 8], the authors also proved that if $\mathcal{I}$ is an ideal such that player II has a winning strategy in $G(\mathcal{I})$ and X is a Polish space, then iterating (finitely many times) the process of taking $\mathcal{I}$-pointwise limits gives the same functions as ordinary limits (i.e. $\mathcal{I}$-pointwise limit of a sequence of functions of the Baire class n is of the Baire class $n+1$ for $n < \omega$). In Theorem 3.2 we extend this result to all Baire classes.

In Sections 4 and 5 we show how to use this combinatorial description of a winning strategy for player II to prove counterparts of the above results for discrete and equal convergence (Theorems 4.3 and 5.5).

By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\omega)$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals of sets of integers. In particular, an ideal $\mathcal{I}$ is *Borel* (*analytic*) if it is a Borel subset of the Cantor space (if it is a continuous image of a G_δ subset of the Cantor space, respectively).

By [9, Prop. 3], the game $G(\mathcal{I})$ is determined for every Borel ideal $\mathcal{I}$. Moreover, by [9, Lem. 2], player I has a winning strategy if and only if $\mathcal{I}$ contains an isomorphic copy (see definitions in the next section) of ideal

$$\mathrm{Fin} \times \mathrm{Fin} = \big\{A \subset \omega \times \omega\colon\ (\exists N \in \omega)\ (\forall n > N)\ \big\{k\colon\ (n,k) \in A\big\} \text{ is finite}\big\}.$$

Using this characterization it was proved in [9] that for any uncountable Polish space X and a Borel ideal $\mathcal{I}$ either $\mathcal{I}$ contains an isomorphic copy of $\mathrm{Fin} \times \mathrm{Fin}$, or every $\mathcal{I}$-pointwise limit of a sequence of continuous real-valued functions on X is of the first Baire class. In Section 6 we extend this result to various kinds of ideal convergence and higher Baire classes.

Debs and Saint Raymond proved [5, Cor. 7.7] that if $\mathcal{I}$ is an analytic ideal which does not contain isomorphic copy of the ideal $\mathrm{Fin} \times \mathrm{Fin}$ and X is a Polish space, then every $\mathcal{I}$-pointwise limit of a sequence of continuous real-valued functions on X is of the first Baire class. In Section 7 we generalize this result to all Baire classes (Theorem 7.2).

2. Preliminaries

In the sequel, we assume all functions to be real-valued functions defined on a set X.

2.1. Three kinds of convergence

Let $f_n : X \to \mathbb{R}$ $(n \in \omega)$. We say that $(f_n)_{n\in\omega}$ converges to $f : X \to \mathbb{R}$ (see [2]):

- *pointwise* ($\lim f_n = f$) if $\{n \in \omega\colon\ |f_n(x) - f(x)| \geqslant \varepsilon\}$ is finite for every $x \in X$ and $\varepsilon > 0$;
- *discretely* ($\text{d-}\lim f_n = f$) if $\{n \in \omega\colon\ f_n(x) \neq f(x)\}$ is finite for every $x \in X$;
- *equally* ($\text{e-}\lim f_n = f$) if there is a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and $\{n \in \omega\colon\ |f_n(x) - f(x)| \geqslant \varepsilon_n\}$ is finite for every $x \in X$.

For a family $\mathcal{E} \subset \mathbb{R}^X$ of functions we define $\mathrm{LIM}(\mathcal{E})$ ($\text{d-LIM}(\mathcal{E})$ and $\text{e-LIM}(\mathcal{E})$) to be the family of all pointwise (discrete and equal, respectively) limits of sequences of functions from $\mathcal{E}$.

2.2. Associated Baire classes

For a topological space X, *Baire classes* $\mathcal{B}_\alpha(X)$, *discrete Baire classes* $\mathcal{B}^{(d)}_\alpha(X)$ and *equal Baire classes* $\mathcal{B}^{(e)}_\alpha(X)$ are defined in the following way (see [2]): $\mathcal{B}_0(X) = \mathcal{B}^{(d)}_0(X) = \mathcal{B}^{(e)}_0(X) = \mathcal{C}(X)$, where $\mathcal{C}(X)$ is the family of all continuous functions, and for $0 < \alpha < \omega_1$:

- $\mathcal{B}_\alpha(X) = \mathrm{LIM}(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(X))$,
- $\mathcal{B}^{(d)}_\alpha(X) = \text{d-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(d)}_\beta(X))$,
- $\mathcal{B}^{(e)}_\alpha(X) = \text{e-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(e)}_\beta(X))$.

2.3. Ideal convergence and $\mathcal{I}$-Baire classes

Let $\mathcal{I}$ be an ideal. Let $f_n : X \to \mathbb{R}$ $(n \in \omega)$. We say that $(f_n)_{n\in\omega}$ converges to $f : X \to \mathbb{R}$:

- *$\mathcal{I}$-pointwise* ($\mathcal{I} - \lim f_n = f$) if $\{n \in \omega\colon\ |f_n(x) - f(x)| \geqslant \varepsilon\} \in \mathcal{I}$ for every $x \in X$ and $\varepsilon > 0$;

- *$\mathcal{I}$-discretely* ($\mathcal{I} - \text{d-lim}\, f_n = f$) if $\{n \in \omega\colon\ f_n(x) \neq f(x)\} \in \mathcal{I}$ for every $x \in X$;
- *$\mathcal{I}$-equally* ($\mathcal{I} - \text{e-lim}\, f_n = f$) if there is a sequence $(\varepsilon_n)_n$ such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ and $\{n \in \omega\colon\ |f_n(x) - f(x)| \geqslant \varepsilon_n\} \in \mathcal{I}$ for every $x \in X$.

For a family $\mathcal{E} \subset \mathbb{R}^X$ of functions we define $\mathcal{I} - \mathrm{LIM}(\mathcal{E})$ ($\mathcal{I} - \text{d-LIM}(\mathcal{E})$, and $\mathcal{I} - \text{e-LIM}(\mathcal{E})$) to be the family of all $\mathcal{I}$-pointwise ($\mathcal{I}$-discrete, and $\mathcal{I}$-equal, respectively) limits of sequences of functions from $\mathcal{E}$.

For a topological space X we define *$\mathcal{I}$-Baire classes* $\mathcal{B}^{\mathcal{I}}_\alpha(X)$, *$\mathcal{I}$-discrete Baire classes* $\mathcal{B}^{(\mathcal{I}-d)}_\alpha(X)$ and *$\mathcal{I}$-equal Baire classes* $\mathcal{B}^{(\mathcal{I}-e)}_\alpha(X)$ in the following way: $\mathcal{B}^{\mathcal{I}}_0(X) = \mathcal{B}^{(\mathcal{I}-d)}_0(X) = \mathcal{B}^{(\mathcal{I}-e)}_0(X) = \mathcal{C}(X)$, and for $0 < \alpha < \omega_1$:

- $\mathcal{B}^{\mathcal{I}}_\alpha(X) = \mathcal{I} - \mathrm{LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{\mathcal{I}}_\beta(X))$,
- $\mathcal{B}^{(\mathcal{I}-d)}_\alpha(X) = \mathcal{I} - \text{d-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(\mathcal{I}-d)}_\beta(X))$,
- $\mathcal{B}^{(\mathcal{I}-e)}_\alpha(X) = \mathcal{I} - \text{e-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(\mathcal{I}-e)}_\beta(X))$.

2.4. Characterization of Baire classes by Borel sets

Let $\boldsymbol{\Sigma}^0_1(X)$ be the family of all open subsets of X, $\boldsymbol{\Pi}^0_\alpha(X)$ be the family of complements of sets from $\boldsymbol{\Sigma}^0_\alpha(X)$ ($1 \leqslant \alpha < \omega_1$) and $\boldsymbol{\Sigma}^0_\alpha(X)$ be the family of countable unions of sets from $\bigcup_{\beta<\alpha} \boldsymbol{\Pi}^0_\beta(X)$ ($\alpha > 1$). For every $\alpha < \omega_1$, let $\boldsymbol{\Delta}^0_\alpha(X) = \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X)$.

In the sequel, we will use the following characterizations of pointwise, discrete and equal Baire classes (here we put $\boldsymbol{\Pi}^0_0(X) = \{\emptyset, X\}$).

Theorem 2.1. *Let X be a perfectly normal topological space and $\alpha < \omega_1$.*

(1) *$f \in \mathcal{B}_\alpha(X) \iff f$ is $\boldsymbol{\Sigma}^0_{\alpha+1}(X)$-measurable (see e.g.* [3, *Prop.* 3.14]*).*
(2) *$f \in \mathcal{B}^{(d)}_\alpha(X) \iff$ there is a cover $X = \bigcup_{i\in\omega} X_i$ and continuous functions $g_i : X \to \mathbb{R}$ such that $X_i \in \boldsymbol{\Pi}^0_\alpha(X)$ and $f \restriction X_i = g_i \restriction X_i$ for every $i \in \omega$ (see* [3, *Thm.* 4.4]*).*
(3) *$f \in \mathcal{B}^{(e)}_{\alpha+1}(X) \iff$ there is a cover $X = \bigcup_{i\in\omega} X_i$ and functions $g_i : X \to \mathbb{R}$ such that $X_i \in \boldsymbol{\Pi}^0_{\alpha+1}(X)$, $g_i \in \mathcal{B}_\alpha(X)$ and $f \restriction X_i = g_i \restriction X_i$ for every $i \in \omega$ (see* [4, *Thm.* 3.6]*).*

2.5. ω-Diagonalizable filters

It is easy to see that if $\mathcal{I}^\star$ contains all cofinite sets then $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets if and only if there exists a sequence $(\mathcal{Z}_N)_{N\in\omega}$ such that

(1) $\mathcal{Z}_N \subset [\omega]^{<\omega} \setminus \{\emptyset\}$ for each $N \in \omega$,
(2) $|\{A \in \mathcal{Z}_N\colon\ A \subset F\}| = \omega$ for each $N \in \omega$ and $F \in \mathcal{I}^\star$, and
(3) for each $F \in \mathcal{I}^\star$ there is $\mathcal{Z}_N = \{A_{N,m}\colon\ m \in \omega\}$ such that

$$\forall_{m\in\omega} \quad (A_{N,m} \cap F \neq \emptyset).$$

In the sequel we will always use conditions (1)–(3) when we say that the filter $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets $\mathcal{Z}_N$'s.

2.6. Katětov order

We say that an ideal $\mathcal{J}$ *contains an isomorphic copy of ideal* $\mathcal{I}$ ($\mathcal{I} \sqsubseteq \mathcal{J}$ for short) if there exists a bijection $\sigma : \omega \to \omega$ such that $\sigma^{-1}[A] \in \mathcal{J}$ whenever $A \in \mathcal{I}$.

For ideals $\mathcal{I}$ and $\mathcal{J}$ we write $\mathcal{I} \leqslant_K \mathcal{J}$ if there exists a function $\sigma : \omega \to \omega$ such that $\sigma^{-1}[A] \in \mathcal{J}$ whenever $A \in \mathcal{I}$. This order (in fact it is a preorder) is called *Katětov order* and it was introduced by Katětov [6] to investigate ideal convergence of sequences of continuous functions.

Lemma 2.2. *(Essentially Katětov [7].) Let X be a topological space. If* $\mathrm{Fin} \times \mathrm{Fin} \leqslant_K \mathcal{I}$ *and $1 \leqslant \alpha < \omega_1$, then*

(1) $\mathcal{B}_{\alpha+1}(X) \subset \mathcal{I} - \mathrm{LIM}(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(X)) \subset \mathcal{B}^{\mathcal{I}}_\alpha(X)$,
(2) $\mathcal{B}^{(d)}_{\alpha+1}(X) \subset \mathcal{I} - \text{d-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(d)}_\beta(X)) \subset \mathcal{B}^{(\mathcal{I}-d)}_\alpha(X)$,
(3) $\mathcal{B}^{(e)}_{\alpha+1}(X) \subset \mathcal{I} - \text{e-LIM}(\bigcup_{\beta<\alpha} \mathcal{B}^{(e)}_\beta(X)) \subset \mathcal{B}^{(\mathcal{I}-e)}_\alpha(X)$.

Proof. (1) is proved in [7], and (2) and (3) can be proved the same way. $\square$

3. Pointwise convergence

Lemma 3.1. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then*

$$\mathcal{I}-\mathrm{LIM}\left(\bigcup_{\beta<\alpha}\mathcal{B}_\beta(X)\right)=\mathcal{B}_\alpha(X)$$

for every $1\leqslant\alpha<\omega_1$.

Proof. By $B(y_0,\varepsilon)$ (resp. $\overline{B}(y_0,\varepsilon)$) we denote the open ball $\{y\in\mathbb{R}\colon |y-y_0|<\varepsilon\}$ (the closed ball $\{y\in\mathbb{R}\colon |y-y_0|\leqslant\varepsilon\}$, respectively). Suppose that $f_n : X\to\mathbb{R}$, $f_n\in\mathcal{B}_{\beta_n}$ ($\beta_n<\alpha$ for each n) such that $f=\mathcal{I}-\lim f_n$. Let $\mathcal{Z}_N=\{A_{N,0},A_{N,1},\ldots\}$ ($N\in\omega$) be a family of $\mathcal{I}^\star$-universal sets which ω-diagonalize $\mathcal{I}^\star$.

First we claim that for any x, y and ε the following conditions are equivalent.

(i) $f(x)\in B(y,\varepsilon)$;
(ii) $\exists_{n\in\omega}\ f(x)\in B(y,\varepsilon\cdot(1-\frac{1}{n}))$;
(iii) $\exists_{n'\in\omega}\ \{m\in\omega\colon\ f_m(x)\in\overline{B}(y,\varepsilon\cdot(1-\frac{1}{n'}))\}\in\mathcal{I}^\star$;
(iv) $\exists_{n''\in\omega}\ \{m\in\omega\colon\ f_m(x)\in\overline{B}(y,\varepsilon\cdot(1-\frac{1}{n''}))\}\in\mathcal{I}^+$.

The implications "(i) $\Rightarrow$ (ii)" and "(iii) $\Rightarrow$ (iv)" are obvious. The implication "(ii) $\Rightarrow$ (iii)" follows from the fact that if we fix n with $f(x)\in B(y,\varepsilon\cdot(1-1/n))$ and take $\delta>0$ such that

$$B\big(f(x),\delta\big)\subset B\left(y,\varepsilon\cdot\left(1-\frac{1}{n}\right)\right)\subset\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n}\right)\right),$$

then since $(f_m(x))_m$ is $\mathcal{I}$-convergent to $f(x)$,

$$\left\{m\in\omega\colon\ f_m(x)\in\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n}\right)\right)\right\}\supset\big\{m\in\omega\colon\ f_m(x)\in B\big(f(x),\delta\big)\big\}\in\mathcal{I}^\star.$$

To see the implication "(iv) $\Rightarrow$ (i)" fix n'' such that

$$\left\{m\in\omega\colon\ f_m(x)\in\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n''}\right)\right)\right\}\in\mathcal{I}^+$$

and assume that $f(x)\notin B(y,\varepsilon)$. Then there exists $\delta>0$ such that

$$B\big(f(x),\delta\big)\cap\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n''}\right)\right)=\emptyset.$$

But, since $(f_m(x))_m$ is $\mathcal{I}$-convergent to $f(x)$,

$$\big\{m\in\omega\colon\ f_m(x)\in B\big(f(x),\delta\big)\big\}\in\mathcal{I}^\star,$$

and so

$$\left\{m\in\omega\colon\ f_m(x)\in\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n''}\right)\right)\right\}\in\mathcal{I},$$

a contradiction. This finishes the proof of the first claim.

Next we claim that in the following list of conditions on x, y, n and ε each implies the next:

(v) $\{m\in\omega\colon\ f_m(x)\in\overline{B}(y,\varepsilon\cdot(1-\frac{1}{n}))\}\in\mathcal{I}^\star$;
(vi) $\exists_{N\in\omega}\ \forall_{k\in\omega}\ \exists_{l\in A_{N,k}}\ f_l(x)\in\overline{B}(y,\varepsilon\cdot(1-\frac{1}{n}))$;
(vii) $\{m\in\omega\colon\ f_m(x)\in\overline{B}(y,\varepsilon\cdot(1-\frac{1}{n}))\}\in\mathcal{I}^+$.

Indeed, let

$$A=\left\{m\in\omega\colon\ f_m(x)\in\overline{B}\left(y,\varepsilon\cdot\left(1-\frac{1}{n}\right)\right)\right\}.$$

To see the implication "(v) $\Rightarrow$ (vi)" we assume that $A\in\mathcal{I}^\star$. Since $\mathcal{I}^\star$ is ω-diagonalizable by universal sets $\mathcal{Z}_N=\{A_{N,k}\colon k\in\omega\}$, there exists $N\in\omega$ such that A has non-empty intersection with $A_{N,k}$ for all $k\in\omega$. This gives us condition (vi).

To see the implication "(vi) $\Rightarrow$ (vii)" note that since each set $\mathcal{Z}_N$ is universal, for every $F \in \mathcal{I}^\star$ there is k such that $A_{N,k} \subset F$, so $A \cap F \neq \emptyset$, and consequently $A \in \mathcal{I}^+$. This finishes the proof of the second claim.

From both claims it follows that

$$f(x) \in B(y, \varepsilon) \iff \exists_{n\in\omega}\, \exists_{N\in\omega}\, \forall_{k\in\omega}\, \exists_{l\in A_{N,k}} \quad f_l(x) \in \overline{B}\left(y, \varepsilon \cdot \left(1 - \frac{1}{n}\right)\right).$$

Since all sets $A_{N,k}$ are finite,

$$f^{-1}\big(B(y,\varepsilon)\big) = \bigcup_{n\in\omega} \bigcup_{N\in\omega} \bigcap_{k\in\omega} \bigcup_{l\in A_{N,k}} f_l^{-1}\left[\overline{B}\left(y, \varepsilon \cdot \left(1 - \frac{1}{n}\right)\right)\right] \in \left(\boldsymbol{\Pi}^0_\alpha\right)_\sigma = \boldsymbol{\Sigma}^0_{\alpha+1}.$$

Thus $f \in \mathcal{B}_\alpha(X)$. □

Theorem 3.2. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then $\mathcal{B}^{\mathcal{I}}_\alpha(X) = \mathcal{B}_\alpha(X)$ for every $\alpha < \omega_1$.*

Proof. By transfinite induction based on Lemma 3.1. □

4. Discrete convergence

Suppose that $\mathcal{E} \subset \mathbb{R}^X$. Let

$$B(f,g) = \big\{x \in X\colon\ f(x) = g(x)\big\}, \qquad \mathcal{B}(\mathcal{E}) = \big\{B(f,g)\colon\ f,g \in \mathcal{E}\big\}.$$

By $\delta'(\mathcal{E})$ we denote the family

$$\left\{\bigcap_{n\in\omega} \bigcup_{m\in A_n} B_{n,m}\colon\ B_{n,m} \in \mathcal{B}(\mathcal{E}) \text{ and } A_n \text{ is finite for each } n, m\right\}.$$

Note that $\delta'(\mathcal{E})$ is closed under taking finite unions and countable intersections.

Lemma 4.1. *Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. For every family of functions $(f_n)_n \subset \mathcal{E}$ if $\mathcal{I} - \text{d-lim}\, f_n = f$ then there exists a family $(E_n)_n \subset \delta'(\mathcal{E})$ such that $\bigcup_n E_n = X$ and for each n there is m with $f \restriction E_n = f_m \restriction E_n$.*

Proof. Let $\mathcal{Z}_N = \{A_{N,0}, A_{N,1}, \ldots\}$ $(N \in \omega)$ be a family of $\mathcal{I}^\star$-universal sets which ω-diagonalize $\mathcal{I}^\star$. Denote

$$E^a_N = \bigcap_{t,s\in A_{N,a}} B(f_t, f_s) \cap \bigcap_{c\in\omega} \bigcup_{t\in A_{N,a}} \bigcup_{s\in A_{N,c}} B(f_t, f_s).$$

Since $A_{N,m}$ are finite, $E^a_N \in \delta'(\mathcal{E})$ for each N, a. Note also that if $x \in E^a_N$ then

(1) $\forall_{t,s\in A_{N,a}}\ (f_t(x) = f_s(x))$, and
(2) $\forall_{c\in\omega}\ \exists_{t\in A_{N,a}}\ \exists_{s\in A_{N,c}}\ (f_t(x) = f_s(x))$.

First we prove that $f \restriction E^a_N = f_r \restriction E^a_N$ for every $N, a \in \omega$ and any $r \in A_{N,a}$. Fix any $x \in E^a_N$ and $r \in A_{N,a}$. Since $\mathcal{I} - \text{d-lim}_{n\in\omega} f_n(x) = f(x)$, there exists $F \in \mathcal{I}^\star$ such that $f_i(x) = f(x)$ for all $i \in F$. Since $\mathcal{Z}_N$ is $\mathcal{I}^\star$-universal, there exists $c \in \omega$ with $A_{N,c} \subset F$, i.e. $f_s(x) = f(x)$ for all $s \in A_{N,c}$. By (2), one can fix $t \in A_{N,a}$ and $s \in A_{N,c}$ with $f_t(x) = f_s(x)$. But $f_s(x) = f(x)$, and so $f_t(x) = f(x)$. To finish the first part of the proof it is enough to observe that from (1) it follows that $f_r(x) = f_t(x) = f(x)$.

Next we show that $X = \bigcup_{N,a\in\omega} E^a_N$. Fix any $x \in X$. Since $\mathcal{I} - \text{d-lim}_{n\in\omega} f_n(x) = f(x)$, there exists $F \in \mathcal{I}^\star$ such that $f_i(x) = f(x)$ for all $i \in F$. Since $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{Z}_N$'s, there exist $N, a \in \omega$ such that $A_{N,a} \subset F$ and $A_{N,c} \cap F \neq \emptyset$ for all $c \in \omega$. From the definition of E^a_N it follows that $x \in E^a_N$. □

Lemma 4.2. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then*

$$\mathcal{I} - \text{d-LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}^{(d)}_\beta(X)\right) = \mathcal{B}^{(d)}_\alpha(X)$$

for every $1 \leqslant \alpha < \omega_1$.

Proof. Let $\mathcal{E} = \{f_n\colon n \in \omega\} \subset \bigcup_{\beta<\alpha} \mathcal{B}^{(d)}_\beta$ and $\mathcal{I} - \text{d-lim}\, f_n = f$. Since $\mathcal{B}^{(d)}_\beta \subset \mathcal{B}_\beta$, so $\delta'(\mathcal{E}) \subset \boldsymbol{\Pi}^0_\alpha$. By Lemma 4.1 there exist sets $E_n \in \boldsymbol{\Pi}^0_\alpha$ $(n \in \omega)$ such that $f \upharpoonright E_n = f_n \upharpoonright E_n$.

Since $f_n \in \mathcal{B}^{(d)}_{\beta_n}$ $(\beta_n < \alpha)$ for each n, by Theorem 2.1, there are $A^n_k \in \boldsymbol{\Pi}^0_\alpha$ and continuous functions g^n_k such that $f_n \upharpoonright A^n_k = g^n_k \upharpoonright A^n_k$ for every $n, k \in \omega$.

Let $B^n_k = A^n_k \cap E_n \in \boldsymbol{\Pi}^0_\alpha$. Then $f \upharpoonright B^n_k = g^n_k \upharpoonright B^n_k$ for every $n, k \in \omega$. Thus, using again Theorem 2.1, $f \in \mathcal{B}^{(d)}_\alpha$. $\square$

Theorem 4.3. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then $\mathcal{B}^{(\mathcal{I}-d)}_\alpha(X) = \mathcal{B}^{(d)}_\alpha(X)$ for every $\alpha < \omega_1$.*

Proof. By transfinite induction based on Lemma 4.2. $\square$

5. Equal convergence

Lemma 5.1. *(See [3, Thm. 5.8].) Let X be a perfectly normal topological space and $0 < \alpha < \omega_1$. If $f \in \mathcal{B}^{(e)}_\alpha(X)$ then $f^{-1}[A] \in \boldsymbol{\Delta}^0_{\alpha+1}(X)$ for every interval $A \subset \mathbb{R}$ (open, closed, one-side open or closed).*

Lemma 5.2. *(See [1, Thm. 2.4].) Let X be a perfectly normal topological space, $\alpha < \omega_1$ and $A \subset X$. If $f \in \mathcal{B}_\alpha(A)$ then there exist $A^\star \in \boldsymbol{\Pi}^0_{\alpha+2}(X)$, $A^\star \supset A$ and $g \in \mathcal{B}_\alpha(A^\star)$ such that $f = g \upharpoonright A$. Moreover, if $A \in \boldsymbol{\Pi}^0_{\alpha+1}(X)$ then we may assume that $A^\star = X$.*

Lemma 5.3. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then*

$$\mathcal{I} - \text{e-LIM}\big(\mathcal{C}(X)\big) = \mathcal{B}^{(e)}_1(X).$$

Proof. Let $f_n : X \to \mathbb{R}$ $(n \in \omega)$ be continuous and $\mathcal{I} - \text{e-lim}\, f_n = f$. For every $x \in X$, let $F_x \in \mathcal{I}^\star$ be such that $|f_i(x) - f(x)| < \varepsilon_i$ for every $i \in F_x$.

Let $\mathcal{Z}_N = \{A_{N,0}, A_{N,1}, \ldots\}$ $(N \in \omega)$ be a family of $\mathcal{I}^\star$-universal sets which ω-diagonalize $\mathcal{I}^\star$.

For every $N \in \omega$, we put

$$X_N = \big\{x \in X\colon \forall_{m\in\omega}\, \exists_{i\in A_{N,m}}\, \big(\big|f_i(x) - f(x)\big| < \varepsilon_i\big)\big\}.$$

It is not difficult to see that $X = \bigcup_{N\in\omega} X_N$.

Let $Y_N = \mathrm{cl}(X_N)$. We will show that $f \upharpoonright Y_N$ is continuous. Then, by Theorem 2.1, $f \in \mathcal{B}^{(e)}_1(X)$.

Fix any $y \in Y_N$. First of all, we show that $f \upharpoonright (X_N \cup \{y\})$ is continuous at y.

Let $\varepsilon > 0$. Let $M \in \omega$ be such that $\varepsilon_i < \varepsilon/3$ for every $i > M$. Let $m \in \omega$ be such that $A_{N,m} \subset F_y \setminus \{0, 1, \ldots, M\}$. (Then $|f_i(y) - f(y)| < \varepsilon_i$ for every $i \in A_{N,m}$.) Let $U \subset X$ be an open neighborhood of y such that

$$\big|f_i(x) - f_i(y)\big| < \varepsilon/3$$

for every $x \in U$ and $i \in A_{N,m}$.

Let $x \in U \cap X_N$. Since $x \in X_N$, there is $i \in A_{N,m}$ with $|f_i(x) - f(x)| < \varepsilon_i$. Then

$$\big|f(x) - f(y)\big| \leqslant \big|f(x) - f_i(x)\big| + \big|f_i(x) - f_i(y)\big| + \big|f_i(y) - f(y)\big| < \varepsilon_i + \varepsilon/3 + \varepsilon_i < \varepsilon.$$

Since $f \upharpoonright (X_N \cup \{y\})$ is continuous for every $y \in Y_N$, oscillation of $f \upharpoonright X_N$ vanishes at every $y \in Y_N$. Thus $f \upharpoonright X_N$ extends to a continuous function defined on Y_N. Moreover, this extension equals $f \upharpoonright Y_N$. $\square$

Lemma 5.4. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then*

$$\mathcal{I} - \text{e-LIM}\big(\mathcal{B}^{(e)}_\alpha(X)\big) = \mathcal{B}^{(e)}_{\alpha+1}(X)$$

for every $0 < \alpha < \omega_1$.

Proof. Let $f_n : X \to \mathbb{R}$ $(n \in \omega)$, $f_n \in \mathcal{B}^{(e)}_\alpha(X)$ be such that $\mathcal{I} - \text{e-lim}\, f_n = f$. For every $x \in X$, let $F_x \in \mathcal{I}^\star$ be such that $|f_n(x) - f(x)| < \varepsilon_n$ for every $n \in F_x$.

Let $\mathcal{Z}_N = \{A_{N,0}, A_{N,1}, \ldots\}$ $(N \in \omega)$ be a family of $\mathcal{I}^\star$-universal sets which ω-diagonalize $\mathcal{I}^\star$.

For every $N \in \omega$, we put

$$X_N = \big\{x \in X\colon \forall_{m\in\omega}\, \forall_{k\in\omega}\, \exists_{i\in A_{N,m}}\, \exists_{j\in A_{N,k}}\, \big|f_i(x) - f_j(x)\big| < \varepsilon_i + \varepsilon_j\big\}.$$

We claim that:

(1) $X = \bigcup_{N\in\omega} X_N$;
(2) $X_N \in \boldsymbol{\Pi}^0_{\alpha+1}(X)$;
(3) $f \restriction X_N \in \mathcal{B}_\alpha(X_N)$.

Suppose that (1)–(3) are fulfilled. For each N, by Lemma 5.2, there is a function $g_N : X \to \mathbb{R}$ such that $g_N \in \mathcal{B}_\alpha(X)$ and $f \restriction X_N = g_N \restriction X_N$. Then, by Theorem 2.1, $f \in \mathcal{B}^{(e)}_{\alpha+1}(X)$.

Now, we show that (1)–(3) are indeed fulfilled.

(1). Let $x \in X$. Then there is $N \in \omega$ such that $A_{N,m} \cap F_x \neq \emptyset$ for every $m \in \omega$. Let $m, k \in \omega$. Then there are $i \in A_{N,m}$ and $j \in A_{N,k}$ such that $|f_i(x) - f(x)| < \varepsilon_i$ and $|f_j(x) - f(x)| < \varepsilon_j$. Thus $|f_i(x) - f_j(x)| < \varepsilon_i + \varepsilon_j$, so $x \in X_N$.

(2). By Lemma 5.1,

$$|f_i - f_j|^{-1}\big[[0, \varepsilon_i + \varepsilon_j)\big] = (f_i - f_j)^{-1}\big[(-\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j)\big] \in \boldsymbol{\Delta}^0_{\alpha+1}(X),$$

so

$$X_N = \bigcap_m \bigcap_k \bigcup_{i\in A_{N,m}} \bigcup_{j\in A_{N,k}} (f_i - f_j)^{-1}\big[(-\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j)\big] \in \boldsymbol{\Pi}^0_{\alpha+1}(X).$$

(3). Let $a \in \mathbb{R}$. It is enough to show that $(f \restriction X_N)^{-1}[(-\infty, a]] \in \boldsymbol{\Pi}^0_{\alpha+1}(X_N)$ and $(f \restriction X_N)^{-1}[[a, \infty)] \in \boldsymbol{\Pi}^0_{\alpha+1}(X_N)$. We show the first case and the second one is done in a similar manner.

We claim that

$$(f \restriction X_N)^{-1}\big[(-\infty, a]\big] = \big\{x \in X_N\colon \forall_{m\in\omega}\, \exists_{i\in A_{N,m}}\, \big(f_i(x) \leqslant a + 3\varepsilon_i\big)\big\} \in \boldsymbol{\Pi}^0_{\alpha+1}(X_N).$$

The "$\in$"-part follows from the fact that $f_i^{-1}[(-\infty, a + 3\varepsilon_i]] \in \boldsymbol{\Delta}^0_{\alpha+1}(X)$. Now we show the equality.

"$\subset$". Let $x \in X_N$ be such that $f(x) \leqslant a$. Let $m \in \omega$. Let $M \in \omega$ be such that $\varepsilon_j < \min\{\varepsilon_i\colon\ i \in A_{N,m}\}$ for every $j > M$. Let $k \in \omega$ be such that $A_{N,k} \subset F_x \setminus \{0, 1, \ldots, M\}$. Then there are $i \in A_{N,m}$ and $j \in A_{N,k}$ with $|f_i(x) - f_j(x)| < \varepsilon_i + \varepsilon_j$ and $|f_j(x) - f(x)| < \varepsilon_j$. Thus,

$$f_i(x) < f_j(x) + \varepsilon_i + \varepsilon_j < f(x) + \varepsilon_j + \varepsilon_i + \varepsilon_j < a + 3\varepsilon_i.$$

"$\supset$". Let $x \in X_N$ be such that $\forall_{m\in\omega}\, \exists_{i\in A_{N,m}}\, (f_i(x) \leqslant a + 3\varepsilon_i)$. Suppose, on the contrary, that $f(x) > a$. Let $\varepsilon = f(x) - a > 0$. Let $M \in \omega$ be such that $4\varepsilon_i < \varepsilon$ for every $i > M$. Let $m \in \omega$ be such that $A_{N,m} \subset F_x \setminus \{0, 1, \ldots, M\}$. Let $i \in A_{N,m}$ be such that $f_i(x) \leqslant a + 3\varepsilon_i$. Since $|f_i(x) - f(x)| < \varepsilon_i$, so

$$f(x) < f_i(x) + \varepsilon_i \leqslant a + 4\varepsilon_i < a + \varepsilon = f(x),$$

a contradiction. $\square$

Theorem 5.5. *Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Then $\mathcal{B}^{(\mathcal{I}-e)}_n(X) = \mathcal{B}^{(e)}_n(X)$ for every $n < \omega$.*

Proof. By Lemma 5.3 for $n = 1$, and by induction based on Lemma 5.4 for $1 < n < \omega$. $\square$

Problem 1. Let X be a perfectly normal topological space. Let $\mathcal{I}$ be an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Does $\mathcal{B}^{(\mathcal{I}-e)}_\alpha(X) = \mathcal{B}^{(e)}_\alpha(X)$ for every $\alpha < \omega_1$?

6. $\mathcal{I}$-Baire classes for Borel ideals $\mathcal{I}$

Theorem 6.1 *(Laczkovich–Recław). (See [9].) Let X be an uncountable Polish space and $\mathcal{I}$ be a Borel ideal. Then the following are equivalent*:

(1) *Every $\mathcal{I}$-pointwise limit of a sequence of continuous functions is of the first Baire class.*
(2) *$\mathcal{I}$ does not contain an isomorphic copy of* Fin × Fin.
(3) *$\mathcal{I}$ and $\mathcal{I}^\star$ can be separated by an F_σ set (i.e. there is an F_σ-set $F \subset \mathcal{P}(\omega)$ such that $\mathcal{I}^\star \subset F$ and $\mathcal{I} \cap F = \emptyset$).*
(4) *$\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets.*

Below we extend this theorem to all Baire classes.

Theorem 6.2. *Let X be an uncountable Polish space, let $\mathcal{I}$ be a Borel ideal and $1 \leqslant \alpha < \omega_1$. Then the following are equivalent:*

(1) $\mathrm{Fin} \times \mathrm{Fin} \not\sqsubseteq \mathcal{I}$.
(2) *$\mathcal{I}$ and $\mathcal{I}^\star$ can be separated by an F_σ set.*
(3) *$\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets.*
(4) $\mathcal{B}_\alpha^{\mathcal{I}}(X) = \mathcal{B}_\alpha(X)$.
(5) $\mathcal{B}_\alpha^{(\mathcal{I}-d)}(X) = \mathcal{B}_\alpha^{(d)}(X)$.
(6) $\mathrm{Fin} \times \mathrm{Fin} \not\leqslant_K \mathcal{I}$.

If $\alpha < \omega$ then the above are also equivalent to:

(7) $\mathcal{B}_\alpha^{(\mathcal{I}-e)}(X) = \mathcal{B}_\alpha^{(e)}(X)$.

Proof. "(1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3)". This is Theorem 6.1.

"(3) $\Rightarrow$ (4), (5), (7)". By Theorems 3.2, 4.3 and 5.5.

"$\neg$(6) $\Rightarrow \neg$(4), $\neg$(5), $\neg$(7)". By Lemma 2.2, it is enough to show that $\mathcal{B}_{\alpha+1}(X) \setminus \mathcal{B}_\alpha(X) \neq \emptyset$, $\mathcal{B}_{\alpha+1}^{(d)}(X) \setminus \mathcal{B}_\alpha^{(d)}(X) \neq \emptyset$ and $\mathcal{B}_{\alpha+1}^{(e)}(X) \setminus \mathcal{B}_\alpha^{(e)}(X) \neq \emptyset$.

Whenever A is a subset of X, $\chi_A : X \to \{0,1\}$ will denote its characteristic function.

If $A \in \boldsymbol{\Pi}_{\alpha+1}^0(X) \setminus \boldsymbol{\Sigma}_{\alpha+1}^0(X)$, then $\chi_A \in \mathcal{B}_{\alpha+1}^{(d)}(X) \subset \mathcal{B}_{\alpha+1}^{(e)}(X) \subset \mathcal{B}_{\alpha+1}(X)$ (by Theorem 2.1), and $\chi_A \notin \mathcal{B}_\alpha(X) \supset \mathcal{B}_\alpha^{(e)}(X) \supset \mathcal{B}_\alpha^{(d)}(X)$.

"(6) $\Rightarrow$ (1)" It follows from definitions of "$\leqslant_K$" and "$\sqsubseteq$". □

Remark. By Theorem 7.2 and Lemma 2.2, (1) and (4) in the above theorem are also equivalent if $\mathcal{I}$ is an analytic ideal.

7. $\mathcal{I}$-Baire classes for analytic ideals $\mathcal{I}$

Lemma 7.1. *(Essentially Laczkovich and Recław [9].) Let $\mathcal{I}$ be an ideal such that $\mathcal{B}_1^{\mathcal{I}}(X) = \mathcal{B}_1(X)$ for every Polish space X. Then*

$$\mathcal{I} - \mathrm{LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta(X)\right) = \mathcal{B}_\alpha(X)$$

for every Polish space X and $1 \leqslant \alpha < \omega_1$.

Proof. If α is a successor ordinal then it is [9, Prop. 10]. Assume that α is a limit ordinal.

Let $(X, \mathcal{T})$ be a Polish space. Let $f_n \in \mathcal{B}_{\beta_n}(X, \mathcal{T})$ ($\beta_n < \alpha$) and $f = \mathcal{I} - \lim f_n$. Let $\{U_n\colon n \in \omega\}$ be a basis for the topology on $\mathbb{R}$.

Since $f_n^{-1}[U_k] \in \boldsymbol{\Sigma}_{\beta_n+1}^0(X, \mathcal{T}) \subset \boldsymbol{\Delta}_\alpha^0(X, \mathcal{T})$, so by [8, Thm. 22.18] there is a Polish topology $\mathcal{T}' \supset \mathcal{T}$ such that $\mathcal{T}' \subset \boldsymbol{\Sigma}_\alpha^0(X, \mathcal{T})$ and $f_n^{-1}[U_k] \in \boldsymbol{\Delta}_1^0(X, \mathcal{T}')$ for every $n, k \in \omega$.

Then $f_n \in \mathcal{C}(X, \mathcal{T}')$, hence $f \in \mathcal{B}_1(X, \mathcal{T}')$. Consequently $f^{-1}[U] \in \boldsymbol{\Sigma}_2^0(X, \mathcal{T}') \subset \boldsymbol{\Sigma}_{\alpha+1}^0(X, \mathcal{T})$ for any open $U \subset \mathbb{R}$, so $f \in \mathcal{B}_\alpha(X, \mathcal{T})$. □

Theorem 7.2. *Let X be a Polish space. Let $\mathcal{I}$ be an analytic ideal such that* $\mathrm{Fin} \times \mathrm{Fin} \not\sqsubseteq \mathcal{I}$. *Then $\mathcal{B}_\alpha^{\mathcal{I}}(X) = \mathcal{B}_\alpha(X)$ for every $\alpha < \omega_1$.*

Proof. For $\alpha = 1$ this follows from [5, Cor. 7.7]. Then proceed by transfinite induction based on Lemma 7.1. □

Problem 2. We do not know if the counterparts of Theorem 7.2 for discrete and equal convergence hold. However, in case of equal convergence and finite α, we show that this problem can be reduced to the problem of $\mathcal{I}$-equal convergence of continuous functions (Proposition 7.4).

Lemma 7.3. *Let $\mathcal{I}$ be an ideal such that $\mathcal{B}_1^{(\mathcal{I}-e)}(X) = \mathcal{B}_1^{(e)}(X)$ for every Polish space X. Then*

$$\mathcal{I} - \text{e-LIM}\big(\mathcal{B}_\alpha^{(e)}(X)\big) = \mathcal{B}_{\alpha+1}^{(e)}(X)$$

for every $\alpha < \omega_1$ and Polish space X.

Proof. Let $(X, \mathcal{T})$ be a Polish space. Let $f_n \in \mathcal{B}_\alpha^{(e)}(X, \mathcal{T})$ and $f = \mathcal{I} - \text{e-lim}\, f_n$. Let $\{U_n\colon n \in \omega\}$ be a basis for the topology on $\mathbb{R}$ which consists of open intervals.

By Lemma 5.1, $f_n^{-1}[U_k] \in \mathbf{\Delta}^0_{\alpha+1}(X, \mathcal{T})$, so by [8, Thm. 22.18] there is a Polish topology $\mathcal{T}' \supset \mathcal{T}$ such that $\mathcal{T}' \subset \mathbf{\Sigma}^0_{\alpha+1}(X, \mathcal{T})$ and $f_n^{-1}[U_k] \in \mathbf{\Delta}^0_1(X, \mathcal{T}')$ for every $n, k \in \omega$.

Then $f_n \in \mathcal{C}(X, \mathcal{T}')$, hence $f \in \mathcal{B}_1^{(e)}(X, \mathcal{T}')$.

By Theorem 2.1, there are $X_i \in \mathbf{\Pi}^0_1(X, \mathcal{T}')$ and $g_i \in \mathcal{C}(X, \mathcal{T}')$ $(i \in \omega)$ such that $f \upharpoonright X_i = g_i \upharpoonright X_i$ for every $i \in \omega$. Then $X_i \in \mathbf{\Pi}^0_{\alpha+1}(X, \mathcal{T})$ and $g_i \in \mathcal{B}_\alpha(X, \mathcal{T})$.

Thus, by Theorem 2.1, $f \in \mathcal{B}^{(e)}_{\alpha+1}(X, \mathcal{T})$. $\square$

Induction based on Lemma 7.3 gives us the following proposition.

Proposition 7.4. *Let $\mathcal{I}$ be an ideal such that $\mathcal{B}_1^{(\mathcal{I}-e)}(X) = \mathcal{B}_1^{(e)}(X)$ for any Polish space X. Then $\mathcal{B}_n^{(\mathcal{I}-e)}(X) = \mathcal{B}_n^{(e)}(X)$ for every $n < \omega$ and Polish space X.*

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